

# ON A BOSONIC-PARAFERMIONIC REALIZATION OF $U_q(\widehat{sl}(2))$

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## Abstract

We realize the  $U_q(\widehat{sl}(2))$  current algebra at arbitrary level in terms of one deformed free bosonic field and a pair of deformed parafermionic fields. It is shown that the operator product expansions of these parafermionic fields involve an infinite number of simple poles and simple zeros, which then condensate to form a branch cut in the classical limit  $q \rightarrow 1$ . Our realization coincides with those of Frenkel-Jing and Bernard when the level  $k$  takes the values 1 and 2 respectively.

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# 1 Introduction

The free field realization of quantum affine algebras [1, 2, 3] has attracted a lot of attention in recent years due to both its physical and mathematical applications. Most of the realizations are given in terms of free bosonic fields and are applied either to simply laced quantum affine algebras at level 1, or to  $U_q(\widehat{sl(2)})$  and more recently  $U_q(\widehat{sl(n)})$  at arbitrary level [4, 5, 6, 7]. (See also Ref. [8] for more references and the equivalence of all the bosonic realizations.) The extension of such bosonic realizations to other quantum affine algebras has not yet been achieved and it is believed to be highly complicated.

It is therefore natural to investigate the realizations of quantum affine algebras either in terms of other types of fields such as fermionic, ghost and parafermionic ones, or in terms of a combination of the latter fields and bosonic ones. These realizations might be simpler than the pure bosonic ones and also applicable to more general quantum affine algebras. In fact, a fermionic and ghost realization of  $U_q(\widehat{sl(n)})$  at level zero is achieved in Ref. [9], a mixed bosonic-fermionic realization of  $U_q(\widehat{so(2n+1)})$  at level 1 is derived in Ref. [10], and more recently a pure fermionic realization of  $U_q(\widehat{sl(2)})$  at level 2 has been provided in Ref. [11].

We pursued these investigations and present here a bosonic-parafermionic realization of  $U_q(\widehat{sl(2)})$  at arbitrary level. It is known that in the classical case, in addition to  $U_q(\widehat{sl(2)})$  at arbitrary level, any untwisted non-simply laced affine algebra at level 1 and any twisted simply laced affine algebra at level 2 can be realized with the help of both bosonic and parafermionic fields [12, 13]. As will be seen in this paper, a model for the simplest quantum affine algebra ( $U_q(\widehat{sl(2)})$ ) can be naturally devised by introducing a deformed parafermionic field. We trust that the results reported in this paper will be of use in the construction of realizations of quantum deformations of the aforementioned affine algebras.

We describe in section 2 the  $U_q(\widehat{sl(2)})$  quantum affine algebra at arbitrary level. We then derive in section 3 a bosonic-parafermionic realization of this algebra which is the quantum analogue of the classical bosonic-parafermionic realization of  $\widehat{sl(2)}$  given in Ref. [14].

## 2 The $U_q(\widehat{sl(2)})$ quantum current algebra

The  $U_q(\widehat{sl(2)})$  affine algebra is the unital associative algebra generated by the elements  $\{E_n^\pm, H_m, K^\pm, q^{\pm d}, \gamma^{\pm 1/2}; n \in \mathbb{Z}, m \in \mathbb{Z}_{\neq 0}\}$  that obey the defining relations [15]

$$\begin{aligned}
[H_n, H_m] &= \frac{[2n]}{2n} \frac{\gamma^n - \gamma^{-n}}{q - q^{-1}} \delta_{n+m, 0}, \\
KH_nK^{-1} &= H_n, \\
[H_n, E_m^\pm] &= \pm \sqrt{2} \frac{\gamma^{\mp |n|/2} [2n]}{2n} E_{n+m}^\pm, \\
KE_n^\pm K^{-1} &= q^{\pm 2} E_n^\pm, \\
[E_n^+, E_m^-] &= \frac{\gamma^{(n-m)/2} \Psi_{n+m} - \gamma^{(m-n)/2} \Phi_{n+m}}{q - q^{-1}}, \\
E_{n+1}^\pm E_m^\pm - q^{\pm 2} E_m^\pm E_{n+1}^\pm &= q^{\pm 2} E_n^\pm E_{m+1}^\pm - E_{m+1}^\pm E_n^\pm, \\
q^d E_n^\pm q^{-d} &= q^n E_n^\pm, \\
q^d H_n q^{-d} &= q^n H_n, \\
[\gamma^{\pm 1/2}, x] &= 0, \quad \forall x \in U_q(\widehat{sl(2)}),
\end{aligned} \tag{2.1}$$

with  $[n] \equiv (q^n - q^{-n})/(q - q^{-1})$  and where  $\Psi_n$  and  $\Phi_n$  are given by the mode expansions of the fields  $\Psi(z)$  and  $\Phi(z)$ , which are themselves defined by

$$\begin{aligned}
\Psi(z) &= \sum_{n \geq 0} \Psi_n z^{-n} = K \exp\{\sqrt{2}(q - q^{-1}) \sum_{n > 0} H_n z^{-n}\}, \\
\Phi(z) &= \sum_{n \leq 0} \Phi_n z^{-n} = K^{-1} \exp\{-\sqrt{2}(q - q^{-1}) \sum_{n < 0} H_n z^{-n}\}.
\end{aligned} \tag{2.2}$$

The central element  $\gamma$  acts as the scalar  $q^k$  on the level  $k$  representations of  $U_q(\widehat{sl(2)})$ . For the purpose of constructing field realizations, it is most convenient to re-express this algebra as a quantum current algebra and to define it through the following operator product expansions (OPE's):

$$\begin{aligned}
\Psi(z) \cdot \Phi(w) &= \frac{(z - wq^{2+k})(z - wq^{-2-k})}{(z - wq^{2-k})(z - wq^{-2+k})} \Phi(w) \cdot \Psi(z), \\
\Psi(z) \cdot E^\pm(w) &= q^{\pm 2} \frac{(z - wq^{\mp(2+k/2)})}{z - wq^{\pm(2-k/2)}} E^\pm(w) \cdot \Psi(z), \\
\Phi(z) \cdot E^\pm(w) &= q^{\pm 2} \frac{(z - wq^{\mp(2-k/2)})}{z - wq^{\pm(2+k/2)}} E^\pm(w) \cdot \Phi(z), \\
E^+(z) \cdot E^-(w) &\sim \frac{1}{w(q - q^{-1})} \left\{ \frac{\Psi(wq^{k/2})}{z - wq^k} - \frac{\Phi(wq^{-k/2})}{z - wq^{-k}} \right\}, \quad |z| > |wq^{\pm k}|, \\
E^-(z) \cdot E^+(w) &\sim \frac{1}{w(q - q^{-1})} \left\{ \frac{\Psi(wq^{k/2})}{z - wq^k} - \frac{\Phi(wq^{-k/2})}{z - wq^{-k}} \right\}, \quad |z| > |wq^{\pm k}|, \\
E^\pm(z) \cdot E^\pm(w) &= \frac{(zq^{\pm 2} - w)}{z - wq^{\pm 2}} E^\pm(w) \cdot E^\pm(z).
\end{aligned} \tag{2.3}$$

Here, the quantum currents  $E^\pm(z)$  are the following generating functions of the generators  $E_n^\pm$ :

$$E^\pm(z) = \sum_{n \in \mathbb{Z}} E_n^\pm z^{-n-1}, \quad (2.4)$$

## 2.1 A bosonic-parafermionic realization of $U_q(\widehat{sl}(2))$

Let  $\varphi^\pm(z)$  denote two different deformations of the same bosonic free field and take their mode expansions to be given by

$$\varphi^\pm(z) = \varphi - i\varphi_0 \ln z + ik \sum_{n \neq 0} \frac{q^{\mp|n|k/2}}{[nk]} \varphi_n z^{-n}. \quad (2.5)$$

The operators  $\{\varphi, \varphi_n; n \in \mathbb{Z}\}$  generate a deformed Heisenberg algebra with the following commutation relations:

$$\begin{aligned} [\varphi_n, \varphi_m] &= \frac{[2n][nk]}{2kn} \delta_{n+m,0}, \\ [\varphi, \varphi_0] &= i. \end{aligned} \quad (2.6)$$

Let

$$V^\pm(a, z) \equiv: e^{ia\varphi^\pm(z)}: \quad (2.7)$$

denote a normal ordered vertex operator with  $a$  an arbitrary real number. The bosonic normal ordering symbol  $::$  indicates that for fields between the two colons, the creation modes  $\{\varphi_n, \varphi; n < 0\}$  should be moved to the left of the annihilation modes  $\{\varphi_n; n \geq 0\}$ . Henceforth, we will use the standard convention that operators defined at the same point are understood to be normal ordered. Using this definition of normal ordering, one can derive the following OPE of vertex operators of the exponential type:

$$\begin{aligned} V^\pm(a, z).V^\mp(b, w) &= e^{-ab\langle\varphi^\pm(z)\varphi^\mp(w)\rangle} : V^\pm(a, z)V^\mp(b, w) :, \\ V^\pm(a, z).V^\pm(b, w) &= e^{-ab\langle\varphi^\pm(z)\varphi^\pm(w)\rangle} : V^\pm(a, z)V^\pm(b, w) :, \end{aligned} \quad (2.8)$$

with the vacuum expectation values given by

$$\begin{aligned} \langle \varphi^\pm(z)\varphi^\mp(w) \rangle &= \varphi^\pm(z).\varphi^\mp(w)- : \varphi^\pm(z)\varphi^\mp(w) : \\ &= -\ln z + \frac{k}{2} \sum_{n>0} \frac{[2n]}{n[nk]} w^n z^{-n} \\ &= -\ln z + \frac{k}{2} \sum_{n \geq 0} \ln \left( \frac{1-q^{k+2+2nk}wz^{-1}}{1-q^{k-2+2nk}wz^{-1}} \right), \\ \langle \varphi^\pm(z)\varphi^\pm(w) \rangle &= \varphi^\pm(z).\varphi^\pm(w)- : \varphi^\pm(z)\varphi^\pm(w) : \\ &= -\ln z + \frac{k}{2} \sum_{n>0} \frac{q^{\mp nk}[2n]}{n[nk]} w^n z^{-n} \\ &= -\ln z + \frac{k}{2} \sum_{n \geq 0} \ln \left( \frac{1-q^{2+2nk+k\mp k}wz^{-1}}{1-q^{-2+2nk+k\mp k}wz^{-1}} \right). \end{aligned} \quad (2.9)$$

Here, without loss of generality, we have assumed that  $|q| < 1$ . Our bosonic-parafermionic realization of  $U_q(\widehat{sl(2)})$  at arbitrary level is given by

$$\begin{aligned}
\Psi(z) &= : V^+(\sqrt{\frac{2}{k}}, zq^{k/2})V^-(-\sqrt{\frac{2}{k}}, zq^{-k/2}) : \\
&= q^{\sqrt{2k}\varphi_0} \exp\left(\sqrt{2k}(q - q^{-1}) \sum_{n>0} \varphi_n z^{-n}\right), \\
\Phi(z) &= : V^+(\sqrt{\frac{2}{k}}, zq^{-k/2})V^-(-\sqrt{\frac{2}{k}}, zq^{k/2}) : \\
&= q^{-\sqrt{2k}\varphi_0} \exp\left(-\sqrt{2k}(q - q^{-1}) \sum_{n<0} \varphi_n z^{-n}\right), \\
E^\pm(z) &= \sqrt{[k]}\psi^\pm(z)V^\pm(\pm\sqrt{\frac{2}{k}}, z),
\end{aligned} \tag{2.10}$$

where the ‘basic’ fields  $V^\pm(\pm\sqrt{\frac{2}{k}}, z)$  and  $\psi^\pm(z)$  satisfy the following OPE’s:

$$\begin{aligned}
V^\pm(\pm\sqrt{\frac{2}{k}}, z).V^\mp(\mp\sqrt{\frac{2}{k}}, w) &= z^{-\frac{2}{k}} \frac{(q^{k+2}wz^{-1}; q^{2k})_\infty}{(q^{k-2}wz^{-1}; q^{2k})_\infty} : V^\pm(\pm\sqrt{\frac{2}{k}}, z)V^\mp(\mp\sqrt{\frac{2}{k}}, w) :, \\
V^\pm(\pm\sqrt{\frac{2}{k}}, z).V^\pm(\pm\sqrt{\frac{2}{k}}, w) &= z^{\frac{2}{k}} \frac{(q^{k\mp k-2}wz^{-1}; q^{2k})_\infty}{(q^{k\mp k+2}wz^{-1}; q^{2k})_\infty} : V^\pm(\pm\sqrt{\frac{2}{k}}, z)V^\pm(\pm\sqrt{\frac{2}{k}}, w) :, \\
\psi^\pm(z).\psi^\mp(w) &= z^{\frac{2}{k}} \frac{(q^{k-2}wz^{-1}; q^{2k})_\infty}{(q^{k+2}wz^{-1}; q^{2k})_\infty} \left( \frac{1}{(z-wq^k)(z-wq^{-k})} + \text{regular} \right), \\
\psi^\pm(z).\psi^\pm(w) &= \frac{(wz^{-1})^{\frac{2}{k}}(zq^{\pm 2}-w)}{z-wq^{\pm 2}} \frac{(q^{k\mp k-2}zw^{-1}; q^{2k})_\infty}{(q^{k\mp k+2}zw^{-1}; q^{2k})_\infty} \frac{(q^{k\mp k+2}wz^{-1}; q^{2k})_\infty}{(q^{k\mp k-2}wz^{-1}; q^{2k})_\infty} \psi^\pm(w).\psi^\pm(z), \\
V^\pm(\pm\sqrt{\frac{2}{k}}, z).\psi^\pm(w) &= \psi^\pm(w).V^\pm(\pm\sqrt{\frac{2}{k}}, z) = \text{regular}, \\
V^\pm(\pm\sqrt{\frac{2}{k}}, z).\psi^\mp(w) &= \psi^\mp(w).V^\pm(\pm\sqrt{\frac{2}{k}}, z) = \text{regular},
\end{aligned} \tag{2.11}$$

and where  $(a, q)_\infty$  stands for the infinite product

$$(a, q)_\infty \equiv \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1. \tag{2.12}$$

It can easily be checked that with these OPE’s the quantum currents  $E^\pm(z)$ ,  $\Psi(z)$  and  $\Phi(z)$  as defined by (2.10) do indeed satisfy the  $U_q(\widehat{sl(2)})$  quantum current algebra (2.3). Moreover, let us further verify that the above realization reduces to that of Frenkel and Jing when  $k = 1$  [4], and to that of Bernard when  $k = 2$  [10]. The OPE’s given by (2.11) take the following simple forms when  $k = 1$ :

$$\begin{aligned}
V^\pm(\pm\sqrt{2}, z).V^\mp(\mp\sqrt{2}, w) &= \frac{1}{(z-wq)(z-wq^{-1})} : V^\pm(\pm\sqrt{2}, z)V^\mp(\mp\sqrt{2}, w) :, \\
V^\pm(\pm\sqrt{2}, z).V^\pm(\pm\sqrt{2}, w) &= (z - wq^{1\mp 1})(z - wq^{-1\mp 1}) : V^\pm(\pm\sqrt{2}, z)V^\pm(\pm\sqrt{2}, w) :, \\
\psi^\pm(z).\psi^\mp(w) &= 1 + (z - wq)(z - wq^{-1})(\text{regular}), \\
\psi^\pm(z).\psi^\pm(w) &= \psi^\pm(w).\psi^\pm(z), \\
V^\pm(\pm\sqrt{2}, z).\psi^\pm(w) &= \psi^\pm(w).V^\pm(\pm\sqrt{2}, z) = \text{regular}, \\
V^\pm(\pm\sqrt{2}, z).\psi^\mp(w) &= \psi^\mp(w).V^\pm(\pm\sqrt{2}, z) = \text{regular}.
\end{aligned} \tag{2.13}$$

It is clear from these relations that the parafermionic fields  $\psi^\pm(z)$  can here be identified with the identity operator. The Frenkel-Jing realization

$$\begin{aligned}
\Psi(z) &= : V^+(\sqrt{2}, zq^{1/2}) V^-(-\sqrt{2}, zq^{-1/2}) : \\
&= q^{\sqrt{2}\varphi_0} \exp\left(\sqrt{2}(q - q^{-1}) \sum_{n>0} \varphi_n z^{-n}\right), \\
\Phi(z) &= : V^+(\sqrt{2}, zq^{-1/2}) V^-(-\sqrt{2}, zq^{1/2}) : \\
&= q^{-\sqrt{2}\varphi_0} \exp\left(-\sqrt{2}(q - q^{-1}) \sum_{n<0} \varphi_n z^{-n}\right), \\
E^\pm(z) &= V^\pm(\pm\sqrt{2}, z).
\end{aligned} \tag{2.14}$$

is thus recovered in this case. When  $k = 2$ , the OPE's (2.11) also reduce to the following simple expressions:

$$\begin{aligned}
V^\pm(\pm 1, z).V^\mp(\mp 1, w) &= \frac{1}{(z-w)} : V^\pm(\pm 1, z)V^\mp(\mp 1, w) :, \\
V^\pm(\pm 1, z).V^\pm(\pm 1, w) &= (z - wq^{\mp 2}) : V^\pm(\pm 1, z)V^\pm(\pm 1, w) :, \\
\psi^\pm(z).\psi^\mp(w) &= \frac{z-w}{(z-wq^2)(z-wq^{-2})} + (z-w)(\text{regular}), \\
\psi^\pm(z).\psi^\pm(w) &= -\psi^\pm(w).\psi^\pm(z), \\
V^\pm(\pm 1, z).\psi^\pm(w) &= \psi^\pm(w).V^\pm(\pm 1, z) = \text{regular}, \\
V^\pm(\pm 1, z).\psi^\mp(w) &= \psi^\mp(w).V^\pm(\pm 1, z) = \text{regular}.
\end{aligned} \tag{2.15}$$

These relations now suggest that it is possible to make the following identifications:

$$\psi^\pm(z) \equiv \psi(z), \tag{2.16}$$

where  $\psi(z)$  is a deformed real free fermionic field such that

$$\psi(z).\psi(w) = \frac{z-w}{(z-wq^2)(z-wq^{-2})} + : \psi(z)\psi(w) :, \tag{2.17}$$

with  $: \psi(z)\psi(z) := 0$ . We are here using the (same) symbol  $::$  to denote the fermionic normal ordering. If

$$\psi(z) = \sum_{r \in \mathbb{Z} + 1/2} \psi_r z^{-r-1/2}, \tag{2.18}$$

this normal ordering is defined so as to have the fermionic creation modes  $\{\psi_r, r \leq -1/2\}$  of fields between the two colons put to the left of their annihilation modes  $\{\psi_r, r \geq 1/2\}$ . Up to a trivial scaling factor, this deformed fermionic field is the same as the one introduced by Bernard [10] in his mixed bosonic-fermionic realization of  $U_q(\widehat{so(2n+1)})$  at level 1, and in

particular in the boson-fermion realization of  $U_q(\widehat{sl(2)})$  at level 2, which reads

$$\begin{aligned}
\Psi(z) &= : V^+(1, zq) V^-(-1, zq^{-1}) : \\
&= q^{2\varphi_0} \exp(2(q - q^{-1}) \sum_{n>0} \varphi_n z^{-n}), \\
\Phi(z) &= : V^+(1, zq^{-1}) V^-(-1, zq) : \\
&= q^{-2\varphi_0} \exp(-2(q - q^{-1}) \sum_{n<0} \varphi_n z^{-n}), \\
E^\pm(z) &= \sqrt{[2]} \psi(z) V^\pm(\pm 1, z).
\end{aligned} \tag{2.19}$$

Note that in the limit  $q \rightarrow 1$ , the OPE's of  $\psi^\pm(z)$  and  $\psi^\mp(w)$  given in (2.11) take the form

$$\psi^\pm(z) \cdot \psi^\mp(w) = (z - w)^{\frac{2}{k}} \left( \frac{1}{(z - w)^2} + \text{regular} \right). \tag{2.20}$$

In the quantum case, an infinite number of alternating simple poles and of simple zeros occur in the expansions of  $\psi^\pm(z) \cdot \psi^\mp(w)$ . They are found respectively at  $z = wq^{k+2+2kn}$ ;  $n \geq 0$  and  $z = wq^{k-2+2kn}$ ;  $n \geq 0$ , and when  $q$  is real, they are seen to lay on the segment connecting  $w$  and the origin. It follows from (2.20), that these zeros and poles condensate to form a branch cut in the classical limit  $q \rightarrow 1$ .

It would be interesting to extend this bosonic-parafermionic realization of  $U_q(\widehat{sl(2)})$  to other quantum affine algebras as discussed in the introduction. One might also look at the relation between this realization of  $U_q(\widehat{sl(2)})$  and the bosonic ones. They could be equivalent as in the classical situation (see [16]), owing to the existence of a boson-parafermion correspondence. Let us finally mention that in the classical case, the full parafermionic algebra is generated (see [14, 16]) by a set of fields  $\psi_n^\pm(z)$ ,  $0 \leq n \leq k$ , with  $\psi_n^+(z) \equiv \psi_{k-n}^-(z)$  and  $\psi_0^+(z)$  the identity operator. We have here introduced and made use of only one pair of deformed parafermionic fields (besides the identity operator) that corresponds to the pair  $\psi^\pm(z) \equiv \psi_1^\pm(z)$ . It would be of interest to identify the appropriate  $q$ -analogues of the other  $\psi_n^\pm(z)$  and to study the quantum deformation of the full parafermionic algebra that these deformed operators would generate.

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